Math 1314 – College Algebra
Section 2.6 Other Types of Equations

- Recall: Rules of Exponents: If m, n are natural numbers,
  - $x^m x^n = x^{m+n}$
  - $(xy)^a = x^a y^a$
  - $(x^m)^n = x^{mn}$
  - $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$

$x^2 x^3 = x^5$

$(x\,x)(x\,x\,x)$

$(x^2)^3 = x^6$

$(x^2)(x^2)(x^2)$

$(x\,x)(x\,x)(x\,x)$

- If $a \geq 0$ and n is a natural number, then $a^{1/n}$ (read as the nth root of a) is the nonnegative real number such that $(a^{1/n})^n = a$.
- Rule for Rational Exponents: If m and n are positive integers, the fraction $\frac{m}{n}$ is in lowest terms, and $a^{1/n}$ is a real number, then $a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$

$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$

- If n is a natural number greater than 1 and if $a^{1/n}$ is a real number, then $\sqrt[n]{a} = a^{1/n}$
- In the radical expression $\sqrt[n]{a}$, the symbol $\sqrt{\ }$ is the radical sign, a is the radicand, and n is the index.

- Properties of Radicals
  Let x, y be real numbers and a, b, m, n be natural numbers. Let $\sqrt[n]{x}, \sqrt[n]{y}$ be real numbers.
  - Product Rule: $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$
  - Power Rule: $\sqrt[n]{x^m} = (\sqrt[n]{x})^m$
  - Quotient Rule: $\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}$
  - If n is odd, $\sqrt[n]{x^n} = x$. If n is even, $\sqrt[n]{x^n} = |x|$.

- Power Property of Real Numbers: If a and b are real numbers, n is an integer, and $a = b$, then $a^n = b^n$.

Ex: $x = 3$

Square both sides: $x^2 = 9$

Solve: $\sqrt{x^2} = \pm\sqrt{9}$

$x = \pm 3$

$x = 3$ or $x = -3$ extraneous soln

$3 = 3$ ✓

$-3 = 3$ ✗

NOTE: When we raise both sides of an equation to a power > 1,

MUST check our answers. Can create extraneous solns

Ex: Solve $(x-3)^{3/2} = 64$

$(x-3)^{3/2} = 64$

$\left((x-3)^{3/2}\right)^{1/3} = (64)^{1/3}$

means what expression do we raise to the 3rd power to get 64?

$1^3 = 1$

$2^3 = 2\cdot 2\cdot 2 = 8$

$3^3 = 3\cdot 3\cdot 3 = 27$

$4^3 = 4\cdot 4\cdot 4 = 64$

$\left(\frac{3}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{2}$

$(x-3)^{1/2} = 4$

$\left(\frac{1}{2}\right)\left(\frac{2}{1}\right) = 1$

$\left((x-3)^{1/2}\right)^{2} = (4)^2$

$x - 3 = 16$

$+3 \quad +3$

$x = 19$

Check: $x = 19$: $(19-3)^{3/2} \stackrel{?}{=} 64$

$(16)^{3/2} \stackrel{?}{=} 64$

$(16^{1/2})^3 \stackrel{?}{=} 64$

$4^3 = 64$ ✓

or $(16^3)^{1/2}$

$(4096)^{1/2} = 64$

Ex: Solve $2x^3 + 3x^2 - 2x = 0$.

$a=2 \quad b=3 \quad c=-2$

$ac=-4$

$-1 + 4 = +3$

$-2 + 2 = 0$

3rd sign is −
- signs in ( ) are diff
- Subtract to get +3

bigger # is +

depressed eqn

$x(2x^2 + 3x - 2) = 0$

$x[2x^2 - x + 4x - 2] = 0$

$x[x(2x-1) + 2(2x-1)] = 0$

$x(2x-1)(x+2) = 0$

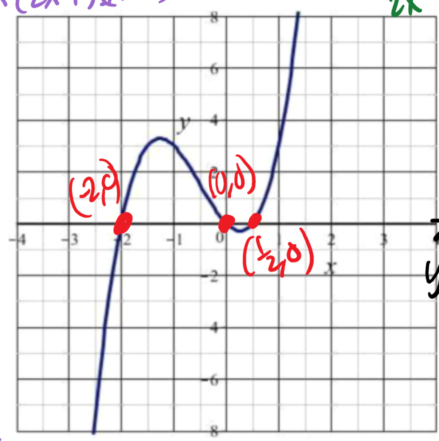
$x=0$ or $2x-1=0$ or $x+2=0$

$2x=1$

$x=\frac{1}{2}$ or $x=-2$

Guide: $2x^2+3x-2=(2x-1)(x+2)$

$y = 2x^3 + 3x^2 - 2x$



Ex: Solve $2m^{2/3} + 3m^{1/3} - 2 = 0$.

$2m^{1/3}m^{1/3} + 2m^{1/3}(2) - 1(m^{1/3}) - 1(2)$

$2m^{2/3} + 4m^{1/3} - m^{1/3} - 2$

$m^{1/3}m^{1/3} = m^{2/3}$

Let $x = m^{1/3}$

$(2m^{1/3}-1)(m^{1/3}+2)=0$

$2m^{1/3}-1=0$ or $m^{1/3}+2=0$

$2m^{1/3}=1$ $\qquad m^{1/3}=-2$

$m^{1/3}=\frac{1}{2}$ $\qquad (m^{1/3})^3=(-2)^3$

$(m^{1/3})^3=\left(\frac{1}{2}\right)^3$ $\qquad m=-8$

$m=\frac{1}{8}$

Check our answers: $2m^{2/3}+3m^{1/3}-2=0$

$m=\frac{1}{8}$: $2\left(\frac{1}{8}\right)^{2/3}+3\left(\frac{1}{8}\right)^{1/3}-2 \overset{?}{=} 0$

$2\left[\left(\frac{1}{8}\right)^{1/3}\right]^2+3\left(\frac{1}{8}\right)^{1/3}-2 \overset{?}{=} 0$

$2\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)-2 \overset{?}{=} 0$

$\frac{2}{1}\left(\frac{1}{4}\right)+\frac{3}{2}-2 \overset{?}{=} 0$

$\frac{1}{2}+\frac{3}{2}-2 \overset{?}{=} 0$

$\frac{4}{2}-2 \overset{?}{=} 0$

$2-2=0$ ✓

$m=-8$: $2((-8)^{1/3})^2+3(-8)^{1/3}-2 \overset{?}{=} 0$

$2(-2)^2+3(-2)-2 \overset{?}{=} 0$

$2(4)-6-2 \overset{?}{=} 0$

$8-6-2 \overset{?}{=} 0$

$0=0$ ✓

Ex: Solve $\sqrt{x-3}+4=7$

Isolate the radical

$\sqrt{x-3}=3$

$(\sqrt{x-3})^2=(3)^2$

$((x-3)^{1/2})^2=(3)^2$

$x-3=9$

$x=12$

Check: $\sqrt{12-3}+4 \overset{?}{=} 7$

$\sqrt{9}+4 \overset{?}{=} 7$

$3+4=7$ ✓

Ex: Solve $\sqrt{2m+1} = -(1-m)$.

Square both sides:

$(\sqrt{2m+1})^2 = (-(1-m))^2$

$(ab)^m = a^m b^m$

$(-1)^2(1-m)^2$

$2m+1 = +1(1-m)^2$

$2m+1 = (1-m)(1-m)$

$2m+1 = 1 - 2m + m^2$

$-2m-1 \qquad -1-2m$

$0 = m^2 - 4m$

$0 = m(m-4)$

$m=0$ or $m-4=0$

$m=4$

extraneous soln

Check $\sqrt{2m+1} = -1(1-m)$

$m=0$: $\sqrt{2(0)+1} \overset{?}{=} -1(1-0)$

$\sqrt{1} \overset{?}{=} -1(1)$

$1 \overset{?}{=} -1$ ✗

$m=4$: $\sqrt{2(4)+1} \overset{?}{=} -1(1-4)$

$\sqrt{9} \overset{?}{=} -1(-3)$

$3 \overset{?}{=} +3$ ✓

Soln: $m=4$

Ex: Solve $\sqrt{2x+1} = 6 - \sqrt{x+5}$.

Sq. both sides:

$(\sqrt{2x+1})^2 = (6 - \sqrt{x+5})^2$

$(6-\sqrt{x+5})(6-\sqrt{x+5})$

F O I L

$36 - 6\sqrt{x+5} - 6\sqrt{x+5} + (\sqrt{x+5})^2$

$36 - 12\sqrt{x+5} + x + 5$

$2x+1 = 41 - 12\sqrt{x+5} + x$

Isolate radical

$x - 40 = -12\sqrt{x+5}$

Sq. both sides:

$(x-40)^2 = (-12\sqrt{x+5})^2$

$(ab)^m = a^m b^m$

$(-12)^2(\sqrt{x+5})^2$

$(x-40)(x-40)$

$x^2 - 40x - 40x + 1600$

$x^2 - 80x + 1600 = +144(x+5)$

$x^2 - 80x + 1600 = 144x + 720$

$x^2 - 224x + 880 = 0$

$a = 1$, $b = -224$, $c = 880$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{224 \pm \sqrt{(-224)^2 - 4(1)(880)}}{2(1)}$

$x = \frac{224 \pm \sqrt{50176 - 3520}}{2}$

$x = \frac{224 \pm \sqrt{46656}}{2}$

$x = \frac{224 \pm 216}{2} = \frac{224+216}{2} = 220$ extraneous soln

$= \frac{224 - 216}{2} = 4$ $\boxed{x = 4}$

Check $x = 220$:

$\sqrt{2x+1} = 6 - \sqrt{x+5}$

$\sqrt{2(220)+1} \stackrel{?}{=} 6 - \sqrt{220+5}$

$\sqrt{441} \stackrel{?}{=} 6 - \sqrt{225}$

$21 \stackrel{?}{=} 6 - 15$ ✗

Check $x = 4$

$\sqrt{2(4)+1} \stackrel{?}{=} 6 - \sqrt{4+5}$

$\sqrt{9} \stackrel{?}{=} 6 - \sqrt{9}$

$3 = 6 - 3$ ✓

Ex: Solve $p - p^{1/2} = 20$.

$p - p^{1/2} - 20 = 0$

let $p^{1/2} = x$

Compare to: $x^2 - x - 20$

$(x+4)(x-5)$

$+1 \; -20$

$+2 \; -10$

$+4 \; -5$

$(p^{1/2} + 4)(p^{1/2} - 5) = 0$

$p^{1/2} + 4 = 0$ $\quad$ $p^{1/2} - 5 = 0$

$p^{1/2} = -4$ $\quad$ $p^{1/2} = 5$

Sq both sides

$(p^{1/2})^2 = (-4)^2$ $\quad$ $(p^{1/2})^2 = (5)^2$

~~$p = 16$~~ $\quad$ $\boxed{p = 25}$

check: $p - p^{1/2} = 20$

$p = 16$: $16 - 16^{1/2} \stackrel{?}{=} 20$

$16 - 4 \stackrel{?}{=} 20$ ✗

$p = 25$: $25 - 25^{1/2} \stackrel{?}{=} 20$

$25 - 5 \stackrel{?}{=} 20$ ✓

- The absolute value of a real number x, denoted $|x|$ is given by:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

$|3| = 3$ $|-4| = 4$ $|0| = 0$ $|-4| = -(-4) = +4$

- The absolute value of a number represents distance b/t 0 & that # on a # line
- Suppose $|x| = 5$. What values can x be? 5 or −5
- Suppose $|x| = 0$. What values can x be? 0
- Suppose $|x| = -7$. What values can x be? None
- So the absolute value of any expression must return a non-negative value.
- **Summarizing:** Suppose $|x| = c$ where c is a real number then:
  - if $c > 0$ then there are two solutions: $x = c$ or $x = -c$
  - if $c = 0$ then there is one solution: $x = 0$
  - if $c < 0$ then there are zero solutions.

Applying this concept we can solve similar equations.

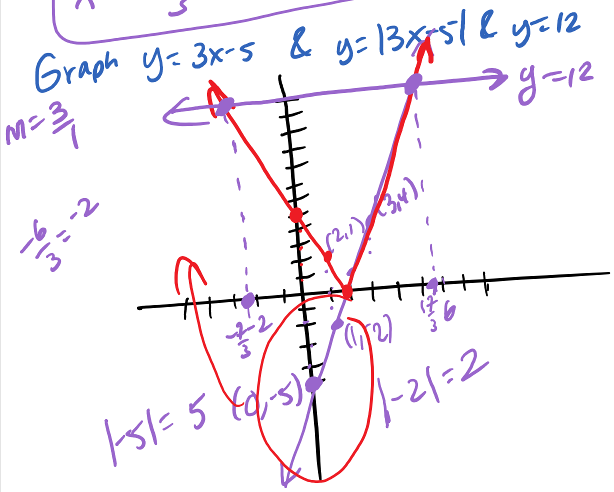
Ex: Solve.

(a) $|3x - 5| = 12$

$3x - 5 = -12$ or $3x - 5 = 12$

$3x = -7$, $x = -\frac{7}{3}$ or $3x = 17$, $x = \frac{17}{3}$

Graph $y = 3x - 5$ & $y = |3x - 5|$ & $y = 12$



(b) $4|x + 1| - 3 = 18$

$4|x+1| = 21$

$|x+1| = \frac{21}{4}$

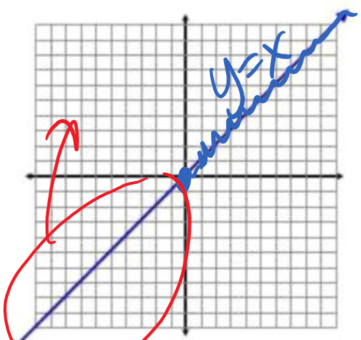
$x + 1 = -\frac{21}{4}$ or $x + 1 = \frac{21}{4}$

$x = -\frac{25}{4}$ or $x = \frac{17}{4}$

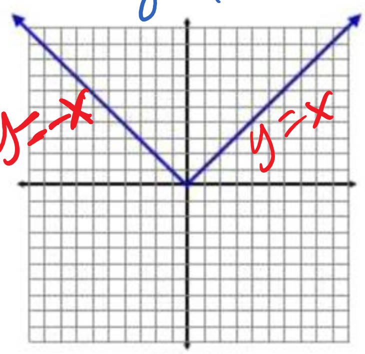
4

(c) $|2x+11| = -5$.

- Consider the graphs of $f(x) = x$ and $g(x) = |x| = \begin{cases} -x & \text{if} \quad x < 0 \\ x & \text{if} \quad x \geq 0 \end{cases}$



$f(x) = x$



$g(x) = |x|$

- How can the the graph of $f(x) = x$ be used to derive the graph of $g(x) = |x|$?

Factor $m^{2/3} + 12\boxed{m^{1/3}} + 36$ Compare to $x^2 + 12\boxed{x} + 36$

$(x+6)(x+6)$

Let $m^{1/3} = x$

$\rightarrow (m^{1/3}+6)(m^{1/3}+6)$

Let $m^{1/5} = x$

$m^{2/5} - 8\boxed{m^{1/5}} + 16$ Compare to $x^2 - 8\boxed{x} + 16$

$(x-4)(x-4)$

$(\boxed{m^{1/5}} - 4)(\boxed{m^{1/5}} - 4)$

$m - 6\boxed{m^{1/2}} + 9$ Compare to $x^2 - 6\boxed{x} + 9$

$(x-3)(x-3)$

Let $m^{1/2} = x$

$(\boxed{m^{1/2}} - 3)(\boxed{m^{1/2}} - 3)$

$s^8 - 22\boxed{s^4} + 121 \Rightarrow$ Compare to $x^2 - 22\boxed{x} + 121$

$(x-11)(x-11)$

$(s^4 - 11)(s^4 - 11)$