Math 1314 – College Algebra
Section 5.5 Zeroes of Polynomial Functions

- Fundamental Theorem of Algebra: If $P(x)$ is a polynomial with positive degree, then $P(x)$ has at least one complex zero.

- Polynomial Factorization Theorem: If $n > 0$ and $P(x)$ is an nth degree polynomial, then $P(x)$ has exactly n factors: $P(x) = a_n(x - r_1)(x - r_2)(x - r_3)\dots(x - r_n)$
$r_1, r_2, r_3, \dots, r_n$ are complex numbers.

- Theorem: If multiple roots are counted individually, the equation $P(x) = 0$ with degree n has exactly n roots among the complex numbers.

- If a factor $x - c$ appears k times, then the zero $x = c$ has multiplicity k.

- Conjugate Pairs Theorem: If an equation $P(x) = 0$ with real coefficients has a complex root $a + bi, (b \neq 0)$, then $a - bi$ (its complex conjugate) is also a zero.

- Rational Zeroes Theorem: Given $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + +a_2x^2 + a_1x + a_0$ with integer coefficients. If the rational root $\frac{p}{q}$ (written in lowest terms) is a zero of $P(x)$, then p is a factor of a_0 and q is a factor of a_n.
Potential rational roots look like: $\frac{\text{factor of constant term}}{\text{factor of leading term}}$

  Ex: $P(x) = (x-5)(x+2)(x-1) = x^3 - 4x^2 - 7x + 10$

- Descartes' Rule of Signs: If $P(x)$ has real coefficients, then the number of positive real zeroes is equal to the number of variations in sign of $P(x)$ or less than that by an even whole number.
The number of negative real zeroes is equal to the number of variations in sign of $P(-x)$ or less than that by an even whole number.

- Rephrase: For potential positive real zeroes: Count sign changes in $P(x)$. Take that number and continue to subtract 2 until you get 0 or 1. The original value and each result of the subtraction could be the number of positive real zeroes.
For potential negative real zeroes: Count sign changes in $P(-x)$. Take that number and continue to subtract 2 until you get 0 or 1. The original value and each result of the subtraction could be the number of negative real zeroes.

- NOTE: To find $P(-x)$, change all signs of coefficients of $x^{\text{odd power}}$

- The Upper and Lower Bounds Theorem: Let P be a polynomial with real coefficients.

  1. If we divide $P(x)$ by $x-b$ (with $b>0$) using synthetic division, and if the row that contains the quotient and remainder has no negative entry, then b is an upper bound for the real zeros of P. No number greater than b can be a root of $P(x)=0$.
  2. If we divide $P(x)$ by $x-a$ (with $a<0$) using synthetic division, and if the row that contains the quotient and remainder has entries that are alternately nonpositive and nonnegative, then a is a lower bound for the real zeros of P. No number less than a can be a root of $P(x)=0$.

- What does this look like?

Ex: Given $P(x)=x^6-7x^5+9x^4+23x^3-50x^2+24x$
(a) List all possible rational zeroes.
(b) Use Descartes' Rule of Signs to find the number of possible positive, negative, and nonreal zeroes.
(c) Is $x=0$ a zero of $P(x)$?
(d) Find all zeroes of $P(x)$.

- Strategy to find all zeroes of a polynomial $P(x)$:

1. Make sure all terms of $P(x)$ are written in descending order.
2. Factor $P(x)$ if possible.
3. Check to see if $x = 0$ is a zero of $P(x)$.
4. Use Descartes' Rule of signs to determine the number of positive, negative, and nonreal roots.
5. Use the Rational Root Theorem to list all possible rational roots.
6. Use synthetic division to find a root. (Remember to write 0 in place of any missing terms).
7. Rewrite $P(x)$, using the root to write as a product of factors.
8. Look at the depressed equation to see if some rational roots can be eliminated. Use this and results from Descartes' Rule to choose the next potential zero to try.
9. Continue until $P(x)$ is written as a product of linear factors and one quadratic factor (i.e. depressed equation is quadratic). Solve the quadratic equation by factoring, completing the square, or quadratic formula.

Ex: Find all zeroes of $P(x) = 2x^4 - x^3 - 2x^2 - 4x - 40$.