Math 1314 – College Algebra
Section 5.6 Rational Functions

- A rational function is a function of the form $y = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
- Always check to see if we can factor the numerator and denominator. If there are any terms in common in the numerator and denominator, we will have a hole in the graph.
- Vertical Asymptote: The line $x = a$ is a VA of the graph of $y = f(x)$ if the y values of the function approach ∞ or $-\infty$ as x approaches a.
- A good place to start looking for VA is where the function is not defined. NOTE: Not all values of x where the function is not defined will be VAs.
- Horizontal Asymptote: The line $y = b$ is a HA of the graph of $y = f(x)$ if the y values of the function approach b as x gets very large and positive (approaches ∞) or as x gets very large and negative (approaches $-\infty$).
- To find horizontal asymptotes – Use long division or divide by the highest power of x in the denominator.

  Recall: As $x \to \infty, \frac{\#}{x^n} \to 0$ if n is a positive integer. As $x \to -\infty, \frac{\#}{x^n} \to 0$ if n is a positive integer.

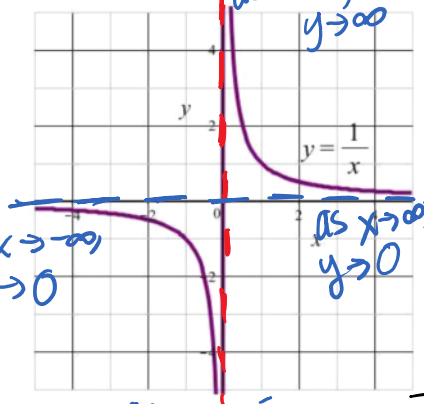
  Ex: Sketch $f(x) = \frac{1}{x}$. Find the domain, range, and any asymptotes.

  Domain: Set denom $\neq 0$: $x \neq 0$ so VA at $x = 0$

  As $x \to -\infty, \frac{1}{x} \to 0$ (below the x-axis)

  As $x \to \infty, \frac{1}{x} \to 0$ (above the x-axis)

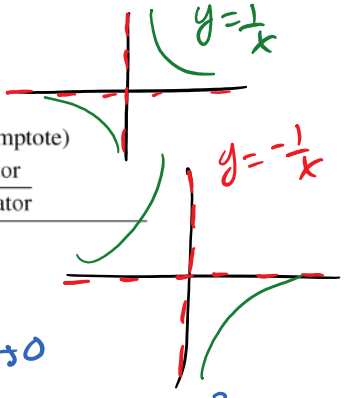
x	y
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{3}$	-3
$\frac{1}{3}$	3
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$



- To find HA of a rational function $y = \frac{P(x)}{Q(x)}$:
  - If degree of numerator < degree of denominator, HA at $y = 0$ (x-axis)
  - If degree of numerator > degree of denominator, no HA. (may have slant or oblique asymptote)
  - If degree of numerator = degree of denominator, HA at $y = \frac{\text{lead coefficient of numerator}}{\text{lead coefficient of denominator}}$

Ex: Find all asymptotes:

(a) $f(x) = \frac{4x+2}{3x-3}$



(b) $g(x) = \dfrac{2x+3}{x^2-5x-14} = \dfrac{(2x+3)}{(x+2)(x-7)}$ Domain: denom $\neq 0$, $x \neq -2$ AND $x \neq 7$. VA: $x=-2$, $x=7$

HA: deg(num)=1, deg(denom)=2. HA at $y=0$

$g(x) = \dfrac{\frac{2x}{x^2}+\frac{3}{x^2}}{\frac{x^2}{x^2}-\frac{5x}{x^2}-\frac{14}{x^2}} = \dfrac{\frac{2}{x}+\frac{3}{x^2}}{1-\frac{5}{x}-\frac{14}{x^2}} = \dfrac{0}{1}$ as $x \to \pm\infty$

(c) $h(x) = \dfrac{x^2+5x}{x-1} = \dfrac{x(x+5)}{(x-1)}$ Domain: $x \neq 1$. VA: $x=1$

HA: deg(num)=2 > deg(denom)=1. NO HA

- Graphing Strategy: Try to factor the numerator and denominator. Find all intercepts (set numerator= 0 for x-intercepts), symmetry, vertical asymptotes (set denominator = 0), and horizontal asymptotes. Plot points.

Ex: Graph $f(x) = \dfrac{3x}{x-2}$.

Factor numerator and denominator. Are there any terms in common?

$f(x) = \dfrac{3x}{x-2} = \dfrac{3x}{(x-2)^1}$ ← odd. No

opposite behavior on either side of VA

Set numerator= 0:

$3x=0$

$x=0$

Thus, x-intercepts are: $(0,0)$

Set denominator= 0:

$x-2=0$

$x=2$

Thus, VA are: $x=2$

Find y-intercept: Set $x=0$

$f(0) = \dfrac{3(0)}{0-2} = \dfrac{0}{-2} = 0$

$(0,0)$

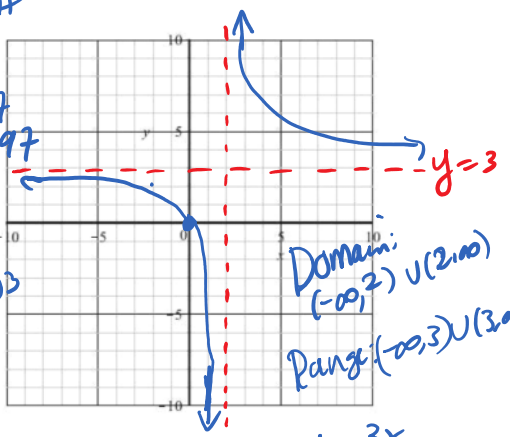
Find HA: deg(num)= 1, deg(denom)=1

use ratio: $\dfrac{3}{1} = 3$

HA at $y=3$

x	y
0	0
1.99	−597
1.999	−5997
2	X
2.01	+603
2.001	+6003

$\frac{0}{\#} = 0$ frac $=0 \Rightarrow$ num $=0$



3 if 1.99 = if (1.99 = 2) = $y = \dfrac{3x}{x-2}$

Solve HA=function to see if graph crosses its HA.

$(x-2)3 = \dfrac{3x}{(x-2)}(x-2)$

$3x-6 \neq 3x$

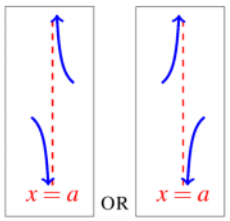
graph does NOT cross its HA.

~~Note: $f(x) = \dfrac{3x}{x-2} = 3\left(\dfrac{x}{x-2}\right)$. Shift graph of $y = \dfrac{1}{x}$ right 2 and stretch vertically by a factor of 3~~

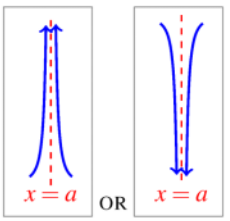
- Regarding the multiplicity of factors in the denominator:

If multiplicity is ODD:	If multiplicity is EVEN:
The graph will have opposite behavior to left and right of the VA where that factor=0. One tail will run to ∞ and the other will run to −∞.	The graph will have same behavior to left and right of the VA where that factor=0. Both tails will run to ∞ or both will run to −∞.

$(x-a)^{odd}$



$(x-a)^{even}$



Ex: Graph $f(x) = \dfrac{3x(x+7)}{(x+7)(x-2)} = \dfrac{3x}{x-2}, x \neq -7$

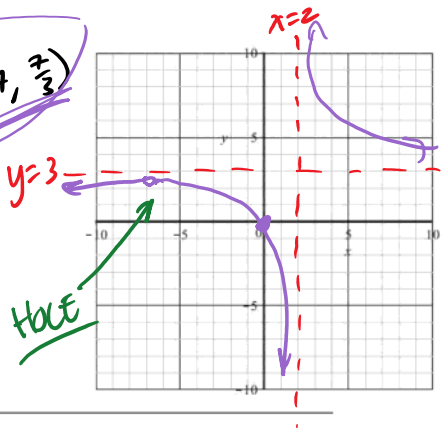
Domain: denom $\neq 0$ $x \neq 2$ $\quad -7$

HOLE: at $x=-7$ y: $\dfrac{3(-7)}{-7-2} = \dfrac{-21}{-9} = \dfrac{21}{9} = \dfrac{7}{3}$

Hole at $(-7, \frac{7}{3})$

VA: $x=2$

HA: $y=3$



- If a function $y = f(x)$ has a vertical asymptote at $x = a$, the graph of $y = f(x)$ will never cross the vertical line $x = a$.
- If a function $y = f(x)$ has a horizontal asymptote at $y = b$, the graph of $y = f(x)$ may cross the horizontal line $y = b$. HA just tells us how the function behaves in the long run.

Ex: Graph $g(x) = \dfrac{3x+6}{x^3-x^2-5x-3} = \dfrac{3x+6}{(x+1)^2(x-3)}$ (Give the basic shape).

Factor numerator and denominator. Are there any terms in common?

NO

$g(x) = \dfrac{3(x+2)}{(x+1)^2(x-3)}$

$y=0$ Frac $=0$

Set numerator $= 0$:

$3(x+2)=0$

$x=-2$

Thus, x-intercepts are: $(-2,0)$

Set denominator $= 0$:

$x=-1$ (even power), $x=3$ (odd power)

Thus, VA are: $x=-1$ $x=3$ (diff. behavior)

Same beh. on either side

Find HA: deg(num) < deg(denom) $y=0$

Find y-intercept:

Set $x=0$

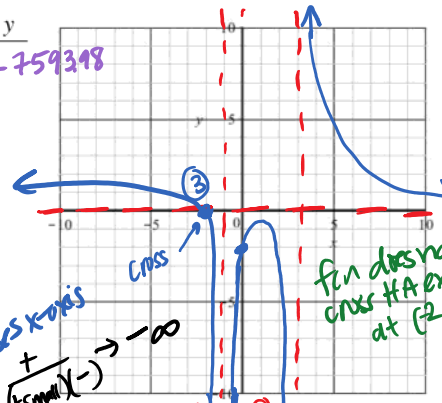
$g(0) = \dfrac{6}{-3} = -2$

$(0,-2)$

x	y
-.99	-759398

OR

x	y
-1000	
-1.01	
-.99	
2.99	
3.01	
1000	



Solve HA=function to see if graph crosses its HA. $y=0$

$0 = \dfrac{3x+6}{(x+1)^2(x-3)}$

$0 = 3x+6$

$-6 = 3x$

$x=-2$

fcn crosses its HA when $x=-2$

y-value is HA.

$(-2,0)$

Ex: Find the domain and graph $f(x) = \dfrac{x^2 - x - 12}{x - 4} = \dfrac{(x-4)(x+3)}{(x-4)} = x+3,\ x \neq 4$

Domain: denom $\neq 0$ $(-\infty, 4) \cup (4, \infty)$
$x \neq 4$

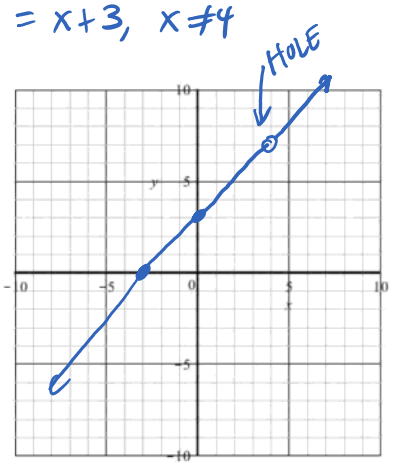
Is $x=4$ a VA? NO.

$x=4$ is a hole

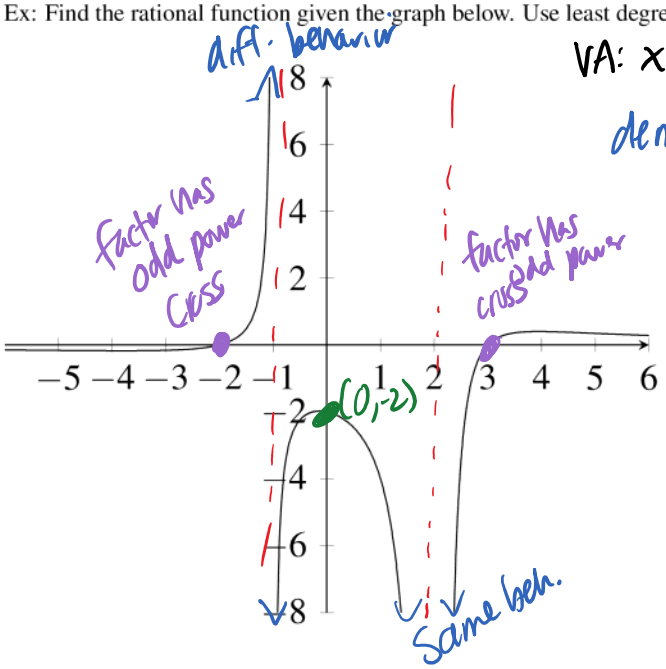
$(4, 7)$

$y = x+3$

$y = 4+3 = 7$



Ex: Find the rational function given the graph below. Use least degree.



VA: $x=-1$, $x=2$

denom: $(x+1)^1(x-2)^2$

x-int: $(3,0)$ & $(-2,0)$

Numerator:

$y = \dfrac{a(x-3)^1(x+2)^1}{(x+1)(x-2)^2}$

How do we find a? use y-int $(0,-2)$

$-2 = \dfrac{a(-3)(2)}{1(-2)^2}$

$-2 = \dfrac{-6a}{4}$

$-8 = -6a$

$\dfrac{-8}{-6} = a$

$+\dfrac{4}{3} = a$

$$f(x) = \frac{4(x-3)(x+2)}{3(x+1)(x-2)^2}$$

Take-home quiz

For you to do:

Sketch $f(x) = \dfrac{1}{x-3}$ and $f^{-1}(x) = \dfrac{3x+1}{x}$ on the same graph. Show all vertical and horizontal asymptotes.