Math 1314 – College Algebra
7.6 Solving Systems with Gaussian Elimination (Gauss-Jordan Elimination)

- A matrix is an ordered rectangular array of numbers.
  Size is m x n

  #of rows ... #of columns

- Denoted by capital letters. Entries are written a_{ij} for any matrix A.

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 6 & 4 \end{bmatrix}$$

Size 2×3

$a_{11} = 1$ (row, col) $a_{21} = 2$

- We can write a system of equations as an augmented matrix (sometimes called a system matrix):

$3x + 5y + 2z = 22$
$6x - 4y + z = 28$
$2x + 3y + 5z = 9$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 22 \\ 6 & -4 & 1 & 28 \\ 2 & 3 & 5 & 9 \end{array}\right]$$

- A matrix is in Row Echelon form when:
  1. First nonzero entry in a row is 1. (called lead unit or leading 1).
  2. 1's are in a stairstep (1 below and to right of previous 1).
  3. Any row consisting of all zeroes is at the bottom (i.e. last row).

Ex: Are these matrices in row echelon form?

(a) $\begin{array}{cc} x & y \end{array}$ $\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

$1x + 0y = 6$
$0x + y = 2$
$0 = 0$

$x = 6$
$y = 2$

(b) $\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 3 & 2 \end{array}\right]$

(c) $\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

$x = 4$
$y = 3$
$z = -7$

(d) $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 6 \\ 1 & 0 & 3 & 2 \end{array}\right]$

(e) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

- A unit column has one 1 and all other entries are 0.
- Row-reduced Echelon form: Has same conditions as row echelon form AND every column with a leading 1 must be a unit column. (RREF)

- When solving a system of equations, we would like our final augmented matrix to be in row-reduced echelon form so we can easily read-off the solutions.

For a system with:

3 equations and 3 variables:

GOAL: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$

2 equations and 2 variables:

GOAL: $\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$

- Row Operations:
  1. Interchange any two rows. $R_i \leftrightarrow R_j$
  2. Replace any row with a nonzero constant multiple of itself. $cR_i \to R_i$
  3. Replace any row with the sum of the nonzero multiple of another row and itself. $R_i + cR_j \to R_j$
- Notice there are no equal signs. These are not equal.
- Process:
  - Go column by column (left to right) and make unit columns.
  - Start with the $1,1$ entry and change that entry to 1.
    - Two options for the $1,1$ entry:
    - Row Op 1 (interchange row 1 with another row that has a 1 in the first column)
    - or Row Op 2 $\left(\dfrac{1}{\text{\# to change}}\right) * (\text{Row to change}) \to (\text{Row to change})$
  - Next, get zeroes in that column to make a unit column.
    - Use Row Op 3:
      (-# to change)*(Row with leading 1) + (Row to change) $\to$ Row to change
  - Move over to the next column to the right. Get the leading 1 in the appropriate entry (in the stairstep).
    - NOTE: Cannot use Row Op 1 for any entry other than the $1,1$. Need to use Row Op 2.
  - Get 0's in that column. Use Row Op 3.
  - Continue until all columns are unit columns (one 1 and all other entries are 0).
- NOTE: This process works every time if you are nervous. You may see other ways to get the matrix into row-reduced echelon form. That is fine, as long as the row operations are done correctly.
- We are using the multiplicative inverse of the number we are trying to change in Row Op 2.
- We are using the additive inverse of the number we are trying to change in Row Op 3. Since we created a leading 1 in that column already, it is convenient to use that to get our 0.

Ex: Solve by Gauss-Jordan elimination:
$3x - 5y = 19$
$x + 2y = -1$

GOAL: $\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$

$\left[\begin{array}{cc|c} 3 & -5 & 19 \\ 1 & 2 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 3 & -5 & 19 \end{array}\right]$

need 3 in 2,1 entry to be a 0

$-3R_1 + R_2 \to R_2$

$-3R_1$	-3	-6	3
$+R_2$	3	-5	19
new R_2	0	-11	22

$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -11 & 22 \end{array}\right] \xrightarrow{-\frac{1}{11} R_2 \to R_2}$

change -11 to 1 in 2,2 entry

$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -2 \end{array}\right] \xrightarrow{-2R_2 + R_1 \to R_1}$

change 2 in 1,2 entry to 0

$-2R_2$	0	-2	4
$+R_1$	1	2	-1
new R_1	1	0	3

$$\begin{array}{ccc} x & y & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

$\boxed{x = 3,\ y = -2}$ exactly one soln

If a_{11} entry is not 1, interchange two rows or use RowOp2
$\left(\frac{1}{\text{\# to change to 1}}\right)(\text{Row to change}) \to (\text{Row to change})$

Change any other entries in column to 0 with RowOp3
$(-\text{\# to change to 0})(\text{Row with the leading 1}) + (\text{Row to change})$
$\to (\text{Row to change})$

Have you successfully pivoted on the a_{11} entry?
Do you have a unit column?

Move over a column and make the leading 1. Use RowOp2
$\left(\frac{1}{\text{\# to change to 1}}\right)(\text{Row to change}) \to (\text{Row to change})$

Change any other entries in column to 0 with RowOp3
$(-\text{\# to change to 0})(\text{Row with the leading 1}) + (\text{Row to change})$
$\to (\text{Row to change})$

Have you successfully pivoted on the leading 1?
Do you have a unit column?

Continue with the other columns until all are unit columns
or the matrix is in row-reduced form
(or row-reduced echelon form).

Read off solutions.

Ex: Solve by Gauss-Jordan elimination:
$2x+y+z=2$
$x+y+2z=4$
$-x-y-3z=-5$

Ex: Solve by Gauss-Jordan elimination:
$3x - 6y = 9$
$-2x + 4y = 6$

Ex: Solve by Gauss-Jordan elimination:
$3x - 6y = -9$
$-4x + 8y = 12$

Ex: $120,000$ gallons of fuel are to be divided between two airlines. Triple A requires twice as much fuel as Unity. How much fuel should be allocated to Triple A?

Ex: Librarians fill shelves completely. One 35 inch shelf can hold 3 dictionaries, 5 atlases, and 1 thesaurus, or 6 dictionaries and 2 thesauruses, or 2 dictionaries, 4 atlases, and 3 thesauruses. How wide is one copy of each book?

t = width of thesaurus
d = width of dictionary
a = width of atlas

$3d + 5a + 1t = 35$
$6d + 2t = 35$
$2d + 4a + 3t = 35$

$t + 3d + 5a = 35$
$2t + 6d + 0a = 35$
$3t + 2d + 4a = 35$

$$\begin{array}{ccc} t & d & a \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 35 \\ 2 & 6 & 0 & 35 \\ 3 & 2 & 4 & 35 \end{array}\right] \cdots \rightarrow \begin{array}{c} \begin{array}{ccc} t & d & a \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{9}{2} \\ 0 & 0 & 1 & \frac{7}{2} \end{array}\right] \end{array}$$

$t = 4$ in $\Rightarrow$ thesaurus is 4 in wide

$d = \frac{9}{2}$ dictionary is $\frac{9}{2}$ 4.5 in wide

$a = \frac{7}{2}$ atlas is $\frac{7}{2} = 3.5$ in wide

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ -1 & -1 & -3 & -5 \end{array}\right] \longrightarrow \cdots \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x = -1$
$y = 3$
$z = 1$

$2x + y + z = 2$
$x + y + 2z = 4$
$-x - y - 3z = -5$

$$\begin{array}{cc} x & y \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 17 \\ 0 & 0 & 4 \end{array}\right]$$

$x = 17$
$0 \neq 4$

$1x + 0y = 17$
$0x + 0y = 4$

NO SOLN