Math 1314 – College Algebra
7.6 Solving Systems with Gaussian Elimination (Gauss-Jordan Elimination)

- A matrix is an ordered rectangular array of numbers.
  Size is m x n

- Denoted by capital letters. Entries are written a_{ij} for any matrix A.

- We can write a system of equations as an augmented matrix (sometimes called a system matrix):

  $3x+5y+2z=22$
  $6x-4y+z=28$
  $2x+3y+5z=9$

- A matrix is in Row Echelon form when:
  1. First nonzero entry in a row is 1. (called lead unit or leading 1).
  2. 1's are in a stairstep (1 below and to right of previous 1).
  3. Any row consisting of all zeroes is at the bottom (i.e. last row).

  Ex: Are these matrices in row echelon form?

  (a) $\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$ (b) $\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 3 & 2 \end{array}\right]$ (c) $\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \end{array}\right]$

  (d) $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 6 \\ 1 & 0 & 3 & 2 \end{array}\right]$ (e) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

- A unit column has one 1 and all other entries are 0.

- Row-reduced Echelon form: Has same conditions as row echelon form AND every column with a leading 1 must be a unit column. (RREF)

- When solving a system of equations, we would like our final augmented matrix to be in row-reduced echelon form so we can easily read-off the solutions.

  For a system with:

  3 equations and 3 variables:

  GOAL: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$

  2 equations and 2 variables:

  GOAL: $\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$

- <u>Row Operations</u>:
  1. Interchange any two rows. $R_i \leftrightarrow R_j$
  2. Replace any row with a nonzero constant multiple of itself. $cR_i \rightarrow R_i$
  3. Replace any row with the sum of the nonzero multiple of another row and itself. $R_i + cR_j \rightarrow R_j$

- Notice there are no equal signs. These are not equal.

- Process:
  - Go column by column (left to right) and make unit columns.
  - Start with the 1, 1 entry and change that entry to 1.
    * Two options for the 1, 1 entry:
    * Row Op 1 (interchange row 1 with another row that has a 1 in the first column)
    * or Row Op 2 $\left(\dfrac{1}{\text{\# to change}}\right) * (\text{Row to change}) \rightarrow (\text{Row to change})$
  - Next, get zeroes in that column to make a unit column.
    * Use Row Op 3:

      (-# to change)*(Row with leading 1) + (Row to change) $\rightarrow$ Row to change
  - Move over to the next column to the right. Get the leading 1 in the appropriate entry (in the stairstep).
    * NOTE: Cannot use Row Op 1 for any entry other than the 1, 1. Need to use Row Op 2.
  - Get 0's in that column. Use Row Op 3.
  - Continue until all columns are unit columns (one 1 and all other entries are 0).

- NOTE: This process works every time if you are nervous. You may see other ways to get the matrix into row-reduced echelon form. That is fine, as long as the row operations are done correctly.

- We are using the multiplicative inverse of the number we are trying to change in Row Op 2.

- We are using the additive inverse of the number we are trying to change in Row Op 3. Since we created a leading 1 in that column already, it is convenient to use that to get our 0.

Ex: Solve by Gauss-Jordan elimination:
$3x - 5y = 19$
$x + 2y = -1$

If a_{11} entry is not 1, interchange two rows or use RowOp2

$$\left(\frac{1}{\text{\# to change to 1}}\right)(\text{Row to change}) \to (\text{Row to change})$$

Change any other entries in column to 0 with RowOp3

$$(-\text{\# to change to 0})\,(\text{Row with the leading 1}) + (\text{Row to change}) \to (\text{Row to change})$$

Have you successfully pivoted on the a_{11} entry?
Do you have a unit column?

Move over a column and make the leading 1. Use RowOp2

$$\left(\frac{1}{\text{\# to change to 1}}\right)(\text{Row to change}) \to (\text{Row to change})$$

Change any other entries in column to 0 with RowOp3

$$(-\text{\# to change to 0})\,(\text{Row with the leading 1}) + (\text{Row to change}) \to (\text{Row to change})$$

Have you successfully pivoted on the leading 1?
Do you have a unit column?

Continue with the other columns until all are unit columns
or the matrix is in row-reduced form
(or row-reduced echelon form).

Read off solutions.

Ex: Solve by Gauss-Jordan elimination:
$2x + y + z = 2$
$x + y + 2z = 4$
$-x - y - 3z = -5$

Ex: Solve by Gauss-Jordan elimination:
$3x - 6y = 9$
$-2x + 4y = 6$

Ex: Solve by Gauss-Jordan elimination:
$3x - 6y = -9$
$-4x + 8y = 12$

Ex: $120,000$ gallons of fuel are to be divided between two airlines. Triple A requires twice as much fuel as Unity. How much fuel should be allocated to Triple A?

Ex: Librarians fill shelves completely. One 35 inch shelf can hold 3 dictionaries, 5 atlases, and 1 thesaurus, or 6 dictionaries and 2 thesauruses, or 2 dictionaries, 4 atlases, and 3 thesauruses. How wide is one copy of each book?