MATH 1324 – FINITE MATHEMATICS
CHAPTER 6 MATHEMATICAL FINANCE

- Simple Interest: Situation where interest is calculated on the original principal only.
  $A = P(1+rt)$ where A is accumulated amount, P is principal, r is interest rate → APR, t = time in years
  $I = Prt$
  Ex: A bank pays simple interest at the rate of 2.6 % per year for certain deposits. If a customer deposits $ 200 and makes no withdrawals for 18 months, what is the total amount in the account at the end of 18 months? What is the interest earned?

Use simple interest fmla: $A = P(1+rt)$

0.2.6% → .026

$\frac{18 \text{ months}}{} \cdot \frac{1 \text{ year}}{12 \text{ months}}$

PEMDAS

$A = ?$
$P = 200$
$r = .026$
$t = \frac{18 \text{ months}}{12 \text{ months}} = 1.5 \text{ yrs}$

$A = 200(1 + .026(1.5))$
$A = 200(1 + .026(\frac{18}{12}))$
$A = \$207.80$

.026 [×] 1.5 [=] [+] 1 [=] [×] 200 [=]

Interest earned: Amt in acct at end − Amt you put in
$207.80 - 200 = \$7.80$

- Compound Interest: Here, earned interest is added to the principal and it earns interest.
  $A = P\left(1+\frac{r}{m}\right)^{mt}$ where m = n = the # of compounding periods per year (how many times interest is calculated per year)

- Principal is sometimes called the present value.

- Accumulated amount is sometimes called the future value.

- Here, there is one deposit.

Ex: Suppose we deposit $ 1500 in an account. How much will be in the account after 3 years if we earn 4.94% interest per year compounded

ONE DEPOSIT ⇒ use cmpd fmla

(a) annually $m=n=1$ (b) semiannually $m=n=2$ (c) quarterly $m=n=4$ (d) monthly $m=n=12$

$A = ?$
$P = 1500$
$r = .0494$
$m = n =$
$t = 3$

Start with (d):
d) $A = 1500\left(1 + \frac{.0494}{12}\right)^{12 \cdot 3}$

.0494 [÷] 12 [=] [+] 1 [=] [^] [(] 12 [×] 3 [)] [=] [×] 1500 [=]

$A = \$1739.09$ interest earned: $1739.09 - 1500 = \$239.09$

c) $A = 1500\left(1 + \frac{.0494}{4}\right)^{4 \cdot 3}$
$A = \$1738.04$ interest: $1738.04 - 1500 = \$238.04$

b) $A = 1500\left(1 + \frac{.0494}{2}\right)^{2 \cdot 3}$ $A = \$1736.49$
Interest: $236.49

a) $A = 1500\left(1 + \frac{.0494}{1}\right)^{1 \cdot 3}$ $A = \$1733.46$ Interest: $233.46

Compound Interest – Detailed Example

Compound Interest: Suppose we deposit \$200 into an account earning 3.1% interest per year, compounded monthly. How much will be in the account in 10 years?

1. Let's explore what is happening in the account first.
   (a) At time $t = 0$, we have \$200 in the account, because this was our initial deposit.
   (b) At the end of Month 1, we have \$200 in the account, plus the interest earned in one month. The calculation is $\$200 + \$200(0.031/12) \approx \$200 + \$0.516667 \approx \$200.52$. We round to two decimal places, because we are dealing with money, but this is an approximation.
   (c) At the end of Month 2, we have \$200 in the account, plus the interest earned on the initial \$200, plus the interest earned on the interest earned in Month 1. (Interest is calculated on all money in the account. Interest earns interest too!) The calculation is $\$200.516667 + \$200.516667(0.031/12) \approx \$200.516667 + \$0.5180 \approx \$201.03$.
   (d) At the end of Month 3, we have \$200 in the account, plus the interest earned on the initial \$200, plus the interest earned on the interest earned in Month 1, plus the interest earned on the interest earned in Month 2. Calculation: $\$201.0346681 + \$201.0346681(0.031/12) \approx \$201.0346681 + \$0.51933956 \approx \$201.55$.

   Here is the monthly account balance for the first year, with values rounded to the nearest cent:

Month	Approximate Amount in account	
0	\$200.00	
1	\$200.52	
2	\$201.03	
3	\$201.55	← See Step (i) above
4	\$202.07	← See Step (ii) above
5	\$202.60	← See Step (iii) above
6	\$203.12	← See Step (iv) above
7	\$203.64	
8	\$204.17	
9	\$204.70	
10	\$205.23	
11	\$205.76	
12	\$206.29	

2. We do not want to continue the calculation this way each month for 10 years (although it is very easy to do using Excel). We will use a formula for the calculation: $A = P\left(1+\frac{r}{m}\right)^{mt}$

VARIABLES:	WORK:	STEPS:
$A = ?$	$A = P\left(1+\frac{r}{m}\right)^{mt}$	
$P =$ 200	$A = 200\left(1+\frac{0.031}{12}\right)^{(12\cdot 10)}$	▪ Fill in our variables.
$r =$ .031		▪ Remembering order of operations, we can do this all at once with our calculator.
$m =$ 12		
$t =$ 10		▪ Calculate $\frac{0.031}{12}$ first.
		▪ Add 1.
		▪ Raise the result to the 120 power.
	$A = \$272.58$	▪ Multiply by 200

- For comparison purposes, the government requires the bank to state their interest rate in terms of an effective interest rate. This is also known as effective yield or annual percentage yield (APY). The effective interest rate gives the actual percentage by which a balance increases in one year.
- Effective Rate of Interest (or annual percentage yield):
  $r_{\text{eff}} = \left(1+\frac{r}{m}\right)^m - 1$
- The effective interest rate is often used to compare accounts that have dfferent compounding intervals and have different stated annual interest rates.

Ex: Compare these accounts and determine which is better for investment and which is better for borrowing.
Account A: 3.62% APR, compounded semiannually and Account B: 3.6% APR, compounded weekly

A: $r_{\text{eff}} = (1+\frac{r}{m})^m - 1$

$r = .0362$
$m = 2$

$r_{\text{eff}} = (1+\frac{.0362}{2})^2 - 1 \approx .036528$

3.6528%
better for borrowing

B: $r = .036$, $m = 52$

$r_{\text{eff}} = (1+\frac{.036}{52})^{52} - 1 \approx .036643$

3.6643%
better for investing

Ex: How much should be deposited in a bank account earning 6.5 % interest per year compounded quarterly so at the end of 5 years there is \$ 2500 in the account?

ONE DEPOSIT ⇒ compound

$A = P(1+\frac{r}{m})^{mt}$

$A = 2500$
$P = ?$
$r = .065$
$m = 4$
$t = 5$

$2500 = P(1+\frac{.065}{4})^{4 \cdot 5}$

.065 ÷ 4 = + 1 = ^ (4 × 5) = 1.38…

2500 ÷ 2nd Ans =

P = \$1811.04

Interest earned:
2500 − 1811.04
\$688.96 in interest

Ex: How long will it take to have \$ 5,500 in an account earning 3.95 % interest per year compounded monthly if we made an initial deposit of \$ 70 ? \$700

ONE DEPOSIT ⇒ compound

$A = P(1+\frac{r}{m})^{mt}$

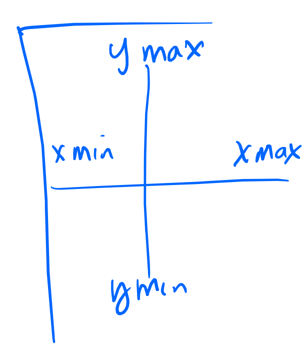
$A = 5500$
$P = 70$
$r = .0395$
$m = n = 12$
$t = ?$

$5500 = 70(1+\frac{.0395}{12})^{12t}$

Y1 = 5500, X2 = $70(1+\frac{.0395}{12})^{12t}$

Deposit \$70
Will take 110.66255
almost 111 years

if we deposit \$700, will take 52.2 yrs



- Annuity: Here, the sequence of payments are made at regular time intervals (terms).
  We will work with ordinary annuities in this course: where the payment is made at the end of the term, the payment period coincides with the interest conversion period, and equal payments are made each term.

$$A = P\left(\frac{\left(1+\frac{r}{m}\right)^{mt}-1}{\frac{r}{m}}\right)$$

- Here, there are multiple deposits.
- You may also see this formula for **future value of an annuity**.
- You may see the term **sinking fund**, which is an account that is set up for a specific purpose at some future date. While you can use a separate formula, we can still use the Annuity formula (because we are saving up money and we are making multiple deposits at regular time intervals that coincide with the interest conversion period).

Ex: Parents deposit $ 200 at the end of every month into a savings account paying 4.65 % interest/year compounded monthly. If they started when the child was 3, how much will be in the account when the child turns 18? How much did they earn in interest?

MULTIPLE DEPOSITS ⇒ ANNUITY $A = P\left(\frac{\left(1+\frac{r}{m}\right)^{mt}-1}{\frac{r}{m}}\right)$

$A = ?$, $P = 200$, $r = .0465$, $n = m = 12$, $t = 18 - 3 = 15$

$A = 200\left(\frac{\left(1+\frac{.0465}{12}\right)^{12 \cdot 15}-1}{\frac{.0465}{12}}\right)$

Parents put in $(200)(12)(15)$ = $36,000

.0465 ÷ 12 = + 1 = ^ (12 × 15) = − 1 = ÷ (.0465 ÷ 12) = × 200 =

$A = \$51{,}923.55$

Interest earned: $51,923.55 − 36,000

Interest: $15,923.55

Ex: Suppose we are saving up to buy a boat so we will need $ 15,000. How much should we deposit into an account earning 3.50 % interest/yr compounded monthly to have the money in 5 years? How much will we earn in interest?

ONE Deposit ⇒ COMPOUND

$A = 15{,}000$, $P = ?$, $r = .0350$, $m = n = 12$, $t = 5$

$15{,}000 = P\left(1+\frac{.0350}{12}\right)^{12 \cdot 5}$

.0350 ÷ 12 = + 1 = ^ (12 × 5) = ; new line: 15000 ÷ 2nd Ans

$P = \$12{,}595.06$

We paid: 12,595.06

Interest earned: 15,000 − 12,595.06 = $2,404.94

Ex: Suppose we are saving up to buy a boat so we will need $ 15,000. How much should we deposit monthly into an account earning 3.50% % interest/yr compounded monthly to have the money in 5 years? How much will we earn in interest?

MULTIPLE DEPOSITS ⇒ ANNUITY

$A = 15{,}000$, $P = ?$, $r = .0350$, $n = m = 12$, $t = 5$

$15{,}000 = P\left(\frac{\left(1+\frac{.0350}{12}\right)^{12 \cdot 5}-1}{\frac{.0350}{12}}\right)$ ≈ 65.466

new line: 15000 ÷ 2nd Ans

$P = \$229.13$

How much did we deposit? $(229.13)(12)(5) = \$13{,}747.80$

Interest earned: 15,000 − 13747.80

Interest: $1252.20

- Amortization: $A = P\left(\frac{1-(1+\frac{r}{m})^{-mt}}{\frac{r}{m}}\right)$

(−)

- Use this formula when paying off a loan. You may also see this formula for **present value of an annuity**.

Ex: A sum of $50,000 is to be repaid over a 5 year period through equal installments made at the end of each year. 8% interest is charged on the unpaid balance and interest is calculated at the end of the year. How much should each installment be so that the loan (principal and interest) is amortized at the end of 5 years?

A=50,000
P=?
r=.08
m=n=1
t=5

$50{,}000 = P\left(\frac{1-(1+\frac{.08}{1})^{-1\cdot 5}}{\frac{.08}{1}}\right)$ P=$12,522.82

3.9927

newline: 50,000 ÷ 2nd Ans

Amortization Table:

Payment period	Interest Charged	Repayment Made	Payment Toward Principal	Outstanding Principal
0	0	0	0	$50,000
1	.08(50,000) $4000	$12,522.82	12,522.82 − 4000 $8522.82	50,000 − 8522.82 $41,477.18
2	.08(41,477.18) $3318.17	$12,522.82	12,522.82 − 3318.17 $9204.65	41,477.18 − 9204.65 $32,272.53
3	.08(32,272.53) $2581.80	$12,522.82	12,522.82 − 2581.80 $9941.02	32,272.53 − 9941.02 $22,331.51
4	.08(22331.51) $1786.52	$12,522.82	12,522.82 − 1786.52 $10,736.30	22,331.51 − 10,736.30 $11,595.21
5	$927.62	~~12,522.82~~ 12,522.83	$11595.20	~~.01~~ 0

Ex: We want to buy a house and found one for $ 182,000. If we can get an interest rate of 2.375% per year compounded monthly for 30 years, and we put 20% of the list price as a down payment, how much will our monthly house payment be (taxes and insurance not included)? How much will we spend in interest?

Down pmt: .20(182,000) = $36,400

A = 182,000 − 36,400 = 145,600
P=?
r=.02375
m=n=12
t=30

$145{,}600 = P\left(\frac{1-(1+\frac{.02375}{12})^{-12\cdot 30}}{\frac{.02375}{12}}\right)$

P=$565.88

At 2.375%, we pay (565.88)(12)(30) $203,716.80 for loan.

Interest: 203,716.80 − 145,600 = $58,116.80

Use 7.1% ⇒ P=$978.48

At 7.1%, we pay (978.48)(12)(30) = 352,252.80

Interest: 352,252.80 − 145,600 = $206,652.80

Ex: In the previous example, what if we financed the house for 20 years instead? 15 years? How much will we pay in interest?

A=145,600
P=?
r=.071
m=n=12
t=20

$145{,}600 = P\left(\frac{1-(1+\frac{.071}{12})^{-12\cdot 20}}{\frac{.071}{12}}\right)$

P=$1137.59

Loan: we pay (1137.59)(12)(20)

$273021.60 − 145,600 = $127,421.60 interest

A=145,600
P=?
r=.071
n=m=12
t=15

$145{,}600 = P\left(\frac{1-(1+\frac{.071}{12})^{-12\cdot 20}}{\frac{.071}{12}}\right)$

P=$1316.85

Loan: Pay (1316.85)(12)(15)

237,033 − 145,600 = $91,433

Ex: You secured a loan to help finance the purchase of your home 5 years ago. The amount of the loan was \$200,000 for a term of 30 years, with interest at the rate of 9%/year compounded monthly. You plan to refinance your mortgage with an interest rate of 4.5%/year compounded monthly.

(a) What was your original monthly payment?

$A = 200{,}000$
$P = ?$
$r = .09$
$m = n = 12$
$t = 30$

$$200{,}000 = P\left(\frac{1-\left(1+\frac{.09}{12}\right)^{-12\cdot 30}}{\frac{.09}{12}}\right)$$

$P = \$1609.25$

(b) What is your outstanding principal after 5 years?

$A = ?$
$P = 1609.25$
$r = .09$
$m = n = 12$
$t = 25$

$$A = 1609.25\left(\frac{1-\left(1+\frac{.09}{12}\right)^{-12\cdot 25}}{\frac{.09}{12}}\right)$$

$A = \$191{,}760.84$

NOTE: we paid $(\$1609.25)(12)(5) = \$96{,}555$

(c) How much equity do you have after 5 years? (Equity=Purchase price - Outstanding principal. NOTE: If housing prices change, Equity=Current Value of Home - Outstanding principal)

Equity $= 200{,}000 - 191{,}760.84 = \$8{,}239.16$

(d) After the rate is reset to 4.5%/year compounded monthly, what will be the new monthly payment? (Round your answer to the nearest cent.)

Refinance

$A = 191{,}760.84$
$P = ?$
$r = .045$
$m = n = 12$
$t = 25$

$$191{,}760.84 = P\left(\frac{1-\left(1+\frac{.045}{12}\right)^{-12\cdot 25}}{\frac{.045}{12}}\right)$$

$P = \$1065.87$

Note: last pmt may be adjusted.

23 yrs → retire at 65 yrs old ⇒ work 42 yrs.
live to be 91

Ex: Suppose we want to retire in 42 years and want \$3500 per month for 26 yrs after we retire. How much should we deposit monthly into an account earning 6.25% interest compounded monthly in order to have enough money for retirement?

Start with RETIREMENT YEARS.

We will have \$X in the account. Interest earns interest and we "pay" ourselves monthly pmts. AMORTIZATION

A = ?
P = 3500
r = .0625
m = n = 12
t = 26

$$A = 3500\left(\frac{1-\left(1+\frac{.0625}{12}\right)^{-12\cdot 26}}{\frac{.0625}{12}}\right)$$

NOTE: We will spend (3500)(12)(26) \$1,092,000

A = \$539,116.15

We need this amount in that account on the day we retire.

WORKING YEARS: Annuity

A = 539,116.15
P = ?
r = .0625
n = m = 12
t = 42

$$539{,}116.15 = P\left(\frac{\left(1+\frac{.0625}{12}\right)^{12\cdot 42}-1}{\frac{.0625}{12}}\right)$$

P = \$ 220.91

We need to deposit \$220.91 into acct every month while working.

How much did we contribute total? (220.91)(12)(42)
\$ 111,338.64

Interest earned: Total amt spent in retirement − Amt we deposited

\$1,092,000 − \$111,338.64 = \$980,661.36