# SECTION 2.4 SETTING UP AND SOLVING SYSTEMS OF LINEAR EQUATIONS

- Matrix Representation:

  1. Line up variables on the left side of the = sign and put constants on the right.
  2. In matrix $A$, put the coefficients (use 0 if missing any variables). Each equation will form a row.
  3. Place the variables in a vertical matrix $X$.
  4. Place constants in a vertical matrix $B$.

Ex: Put the system of equations in matrix representation.

$7+9x-18y=27z$
$43x-8z+36y=57$
$22x+85z=19$

$9x-18y-27z=-7$
$43x+36y-8z=57$
$22x+0y+85z=19$

$$\underset{3\times 3}{\overset{A}{\begin{bmatrix} 9 & -18 & -27 \\ 43 & 36 & -8 \\ 22 & 0 & 85 \end{bmatrix}}} \underset{3\times 1}{\overset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}} \overset{=}{=} \overset{B}{\begin{bmatrix} -7 \\ 57 \\ 19 \end{bmatrix}}$$

- Augmenting matrices:

$$\begin{array}{ccc|c} x & y & z & \\ 9 & -18 & -27 & -7 \\ 43 & 36 & -8 & 57 \\ 22 & 0 & 85 & 19 \end{array}$$

Size 3x4

- We will use Gauss-Jordan Elimination to solve systems of linear equations.

- A matrix is in Reduced row-echelon form when:

  1. Each row of the coefficient matrix consisting entirely of zeroes lies below any other row having nonzero entries.
  2. The first nonzero entry in each row is 1 (called a leading 1)
  3. In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row. (makes a stairstep).
  4. If the column contains a leading 1, then the other entries in that column are zeroes. Called a unit column.

Ex: Is the matrix in reduced row-echelon form?

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 17 \end{array}\right]$

unit column, unit col., unit col

1.✓ 2.✓ 3.✓ 4.✓

RREF form

(b) $\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 1 & 4 \end{array}\right]$

uc, not uc

1.✓ 2.✓ 3.✓ 4.✗

not in RREF form

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 13 \end{array}\right]$

1.✗

not in RREF

(d) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 11 \end{array}\right]$

uc, uc, does not have to be a uc.

1.✓ 2.✓ 3.✓ 4.✓

RREF form

(e) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

uc, uc, does not have to be a uc

1.✓ 2.✓ 3.✓ 4.✓

RREF form

(f) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 19 \\ 0 & 1 & 3 & 2 \end{array}\right]$

uc, uc, does not have to be a uc

1.✓ 2.✓ 3.✓ 4.✓

RREF form

- Row Operations:

  1. Interchange any two rows. $R_i \leftrightarrow R_j$
  2. Replace any row with a nonzero constant multiple of itself. $cR_i \to R_i$
  3. Replace any row with the sum of the nonzero multiple of another row and itself. $R_i + cR_j \to R_i$

- Notice there are no equal signs. These are not equal.

- For a system with:

2 equations and 2 variables:

GOAL: $\begin{array}{cc|c} x & y & \\ 1 & 0 & a \\ 0 & 1 & b \end{array}$ $\quad 1x+0y=a \quad 0x+1y=b \quad \boxed{\begin{matrix} x=a \\ y=b \end{matrix}}$

if $x, y$ are variables, $x=a, y=b$

3 equations and 3 variables:

GOAL: $\begin{array}{ccc|c} x & y & z & \\ 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}$ $\quad x=a \quad y=b \quad z=c$

if $x, y, z$ are variables, $x=a, y=b, z=c$

Ex: Use Gauss-Jordan elimination to solve the system of equations.

$3x+4y=4$

$x-2y=8$

$\left[\begin{array}{cc|c} 3 & 4 & 4 \\ 1 & -2 & 8 \end{array}\right]$

$2\times3$

2nd MATRIX (Hint: Below the MATH button)

→ → EDIT: Input size, then entries.

2nd QUIT

Then 2nd MATRIX → MATH ↓↓↓↓↓ B:rref([A])

2nd + 7 1 2

$\xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \end{array}\right] \quad \boxed{\begin{matrix} x=4 \\ y=-2 \end{matrix}}$ or $(4,-2)$

exactly one soln

Ex: Use Gauss-Jordan elimination to solve the system of equations.

$x+2y-3z=-2$

$3x-y-2z=1$

$2x+3y-5z=-3$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right] \xrightarrow{RREF} \begin{array}{ccc|c} x & y & z & \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}$

$x-z=0$

$y-z=-1$

$0=0$

$x-z=0$

$y-z=-1$

$x=z$

$y=-1+z$ or $y=z-1$

$z=z$

Let $z=t$, any real #

$x=t$

$y=-1+t \quad y=t-1$

$z=t$, any real #

infinitely many solns

$\begin{array}{ccc} x & y & z \\ (t, & -1+t, & t) \end{array}$

$(t, t-1, t)$

Find a particular soln: $t=21$: $(21, 20, 21)$

$t=4$: $(4, 3, 4)$

$t=7$: $(7, 6, 7)$

$t=5$: $(5, 4, 5)$

$t=13$: $(13, 12, 13)$

- Geometrically, solutions lie on a line in 3-D space given by the intersection of the 3 planes determined by the 3 equations.

Ex: Use Gauss-Jordan elimination to solve the system of equations.

$x+y-2z=-3$
$2x-y+3z=7$
$x-2y+5z=0$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 2 & -1 & 3 & 7 \\ 1 & -2 & 5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{7}{3} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Size: $3 \times 4$

$x+\frac{1}{3}z=0$
$y-\frac{7}{3}z=0$
$0 \neq 1$

MATH ▶ Frac

$\boxed{\text{NO SOLN}}$

- Geometrically, two of the three planes Intersect in a straight line but the third plane is parallel to this line of intersection.
- If the number of equations is greater than or equal to the number of variables in a linear system then one of the following is true:
  - The system has exactly one soln
  - The system has Infinitely many solns
  - The system has no soln

If there are fewer equations than variables in a linear system, then the system either has Inf. many solns or no soln.

rows columns

Ex: Determine whether the system has a solution. If so, determine what the solutions are.

3 variables
3 eqns

(a) $\left[\begin{array}{ccc|c} x & y & z & \\ 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -8 \end{array}\right]$

$x=6$
$y=-2$
$z=-8$

exactly one soln
$(6,-2,-8)$

3 variables
4 eqns

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$x=4$
$y=-1$
$z=3$
$0 \neq 1$

NO SOLN

3 variables
2 eqns

(c) $\left[\begin{array}{ccc|c} x & y & z & \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 1 \end{array}\right]$

$x+z=4$
$y=1$

let $z=t$, any real #

$x=\cancel{-z+4}$ $4-z$
$y=1$
$z=z$

Inf. many solns

$x=4-t$
$y=1$
$z=t$

$(4-t,1,t)$ or $(-t+4,1,t)$

Particular soln: $t=3$: $(1,1,3)$
$t=100$ $(-96,1,100)$

2 variables
3 eqns

(d) $\left[\begin{array}{cc|c} x & y & \\ 1 & 0 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$

$x=6$
$y=-2$
$0=0$ ✓

exactly one soln

$x=6$
$y=-2$

$(6,-2)$

ables

4 variables

$x_3+x_4=3$

3 variables
3 eqns

(e) $\begin{array}{ccc|c} x & y & z & \\ 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{array}$

$x=11$
$y=7$
$0=0$ ✓

$x=11$
$y=7$
$z=z$

let $z=t$, any real #
$x=11$
$y=7$
$z=t$

Inf. many Solns.

$(11,7,t)$

A particular soln is: $(11,7,3)$
$(11,7,17)$

4 variables
4 eqns

(f) $\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}$

$x_1=0$
$x_2=8$
$\boxed{x_3+x_4=3}$
$0=0$

$x_3+x_4=3$
$-x_4 \quad -x_4$

Inf many solns

$x_1=0$
$x_2=8$
$x_3=3-x_4$
$x_4=x_4$

Let $x_4=t$, any real #
$x_1=0$
$x_2=8$
$x_3=3-t$
$x_4=t$

$(0,8,3-t,t)$

4 variables
3 eqns

(g) $\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}$

$x_1+x_3=4$
$x_2+x_4=2$
$0=0$ ✓

$x_1=-x_3+4$ or $4-x_3$
$x_2=-x_4+2$ or $2-x_4$
$x_3=x_3$
$x_4=x_4$

Inf. many solns

any real #
let $x_3=t$, $x_4=s$
$x_1=4-t$ or $-t+4$
$x_2=-s+2$ or $2-s$
$x_3=t$
$x_4=s$

$(4-t,2-s,t,s)$

Particular soln:
$t=1$, $s=2$:
$(3,0,1,2)$

$t=1$, $s=0$
$(3,2,1,0)$

Ex: You invested \$30,000 in Bond A and Bond B. If Bond A earned 8% interest and Bond B earned 10% and you earned in total interest \$2640, how much did you invest in each bond?

$a$ = amount invested in Bond A in \$
$b$ = amt invested in Bond B in \$

$.08a+.10b=2640$

$a+b=30{,}000$

$\left[\begin{array}{cc|c} a & b & \\ .08 & .10 & 2640 \\ 1 & 1 & 30{,}000 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} a & b & \\ 1 & 0 & 18{,}000 \\ 0 & 1 & 12{,}000 \end{array}\right]$

$a=18{,}000$
$b=12{,}000$

Size: $2\times3$

Invested \$18,000 in Bond A
\$12,000 in Bond B

Ex: A concert venue has a seating capacity of 900 and charges \$2 for general admission seats, \$3 for balcony seats, and \$4 for reserved admission seats. At a screening with full attendance, there were half as many reserved admission seats as general admission and balcony seats combined. The receipts totaled \$2800. How many general admission seats does the venue have?

video quiz

Ex: Pi's Bakery relies on investments in stocks, bonds, and mutual funds to supplement their annual income. The stocks return 10% per year, the bonds return 8% per year, and the mutual funds return 6% per year. The amount in stocks should be equal to the sum of the amount invested in bonds and three times the amount invested in the mutual funds. If Pi's Bakery invests \$80,000 and needs an annual income of \$7,200 from the investments, how much should be invested in each? Give two specific options.

s = amt invested in stocks

b = amt invested in bonds

m = amt invested in mutual funds

Sum ⇒ add b + 3m

$s = b + 3m$

$.10s + .08b + .06m = 7200$

$s + b + m = 80{,}000$

$s = b + 3m \Longrightarrow s - b - 3m = 0$ or $0 = -s + b + 3m$

$-1 \quad 1 \quad 3 \quad 0$

$$\begin{array}{cccc} s & b & m & \text{Constant} \\ \end{array}$$

$$\left[\begin{array}{ccc|c} .10 & .08 & .06 & 7200 \\ 1 & 1 & 1 & 80{,}000 \\ 1 & -1 & -3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 40{,}000 \\ 0 & 1 & 2 & 40{,}000 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Stock, bond, mf question:

RREF ... $\longrightarrow$

$$\begin{array}{ccc|c} s & b & m & \\ \hline 1 & 0 & -1 & 40{,}000 \\ 0 & 1 & 2 & 40{,}000 \\ 0 & 0 & 0 & 0 \end{array}$$

$s - m = 40{,}000$

$b + 2m = 40{,}000$

$0 = 0$ ✓

$s = 40{,}000 + m$

$b = 40{,}000 - 2m$

$m = m$

Let $m = t$, any real #

$s = 40{,}000 + t$

$b = 40{,}000 - 2t$

$m = t$

$s \geq 0$

$b \geq 0$

$m \geq 0$

$\$0 \leq t \leq \$20{,}000$

Infinitely many solns

$b = 0$: $0 = 40{,}000 - 2t$

$2t = 40{,}000$

$t = 20{,}000$

One option:

$t = 0$: invest \$40,000 in stocks

\$40,000 in bonds

\$0 in mutual funds

$t = \$5000$ invest \$45,000 in stocks

\$30,000 in bonds

\$5,000 in mutual funds

Another option: $t = \$20{,}000$

invest \$60,000 in stocks

\$0 in bonds

\$20,000 in mutual funds