Math 1324 – Finite Mathematics
Section 3.4 Simplex Method

- A linear programming problem consists of a linear objective function to be maximized or minimized subject to certain constraints in the form of linear equations or inequalities.
- The Simplex Method is a method of finding the corner points for a linear programming problem with n variables algebraically.
- In this section, we will only cover standard maximization problems. These problems meet the following conditions:

  $3x - 4y + 2z \le 7$, $x \ge 0$, $y \ge 0$, $z \ge 0$

  1. The objective function is maximized.
  2. All variables in the problem are non-negative. $x \ge 0$, $y \ge 0$, $z \ge 0$ etc
  3. All constraints are of the form: $a_1 x + a_2 y + \ldots + a_n z \le$ a non-negative real number

  variable stuff $\le$ non-neg #

- Here is the Simplex Method for a Standard Maximization Problem:

  1. **Set up the problem, algebraically.** That is, write the objective function to be maximized and the constraints in standard maximization form.
  2. **Convert the linear constraint inequalities into linear equations.** This is done by adding a different slack variable to each inequality, not including non-negativity constraints.
  3. **Construct the initial simplex tableau.** Align variables from each 'equality' and objective function; place the coefficients into a matrix, with the objective function as the bottom row, below a horizontal line.
  4. **Identify the pivot column.** The most negative entry in the bottom row identifies the pivot column.

  $\frac{\text{Constant}}{\text{PC entry}}$

  5. **Identify the pivot row.** Calculate the test ratios for all rows except the bottom row. The ratios are computed by dividing the far right ('constant') column by the corresponding value in the identified column from Step 4. A ratio that has a zero or negative number in the denominator is ignored. The smallest non-negative ratio computed identifies the pivot row. ⊛ NOTE: If 0 is in constants column, $\frac{0}{+\#} = 0$ ← this is the smallest non-neg
  6. **Identify the pivot element.** The entry in the intersection of the pivot column and the pivot row is the pivot element.
  7. **Pivot on the pivot element.** Pivoting can be done using elementary row operations, as previously discussed. However, there are calculator and computer programs which will perform these calculations for you. Often the programs only require the user to input the initial tableau and indicate subsequent pivot elements.
  8. **When there are no negative entries in the bottom row after pivoting, we are finished pivoting and can identify the optimal solution; otherwise, we start again from Step 4.**
  9. **Identify the optimal solution.** Determine basic and non-basic variables. Columns which contain a single 1, with all other entries 0, represent basic variables, and all other columns represent non-basic variables. Set all non-basic variables equal to zero and solve for the basic variables.

- Cross out columns that are not unit columns. Variable = 0
- Read off answers.

Consider $x+y \le 80$

$ $x=10, y=20$: $10+20 \le 80 \Rightarrow 10+20+50=80$

$ $x=5, y=20$: $5+20+55=80$

$ $x=60, y=20$: $60+20+0=80$

Ex: Solve the linear programming problem using the Simplex Method.

Maximize $P = 5x + 3y$

Subject to: $x + y \le 80 \Rightarrow x + y + s_1 = 80$

$3x \le 90 \Rightarrow 3x + s_2 = 90$

$x \ge 0$

$y \ge 0$

Obj fcn: $P = 5x+3y$

$-5x - 3y + P = 0$

- Is this a standard maximization problem?
  1. obj fcn is maximized ✓
  2. variables are non-neg: $x \ge 0, y \ge 0$ ✓
  3. variable stuff $\le$ non-neg # ✓

Initial Simplex Tableau ⟶⟶ Simplex calculator ⟶⟶

x	y	s_1	s_2	P	constant
1	1	1	0	0	80
3	0	0	1	0	90
−5	−3	0	0	1	0

pc

$\frac{80}{1} = 80$

$\frac{90}{3} = 30 \leftarrow$ pr

use smallest ratio

$x = 30$, $y = 50$, $s_1 = 0$, $s_2 = 0$, $P = 300$

An optimal soln has been reached

Final Simplex Tableau

x	y	s_1	s_2	P	constant
0	1	1	$-\frac{1}{3}$	0	50 $y=50$
1	0	0	$\frac{1}{3}$	0	30 $x=30$
0	0	3	$\frac{2}{3}$	1	300 $P=300$

Ex: A department store sells two sizes of televisions: 21 inch and 40 inch. A 21 inch television requires 6 cubic feet of storage space, and takes 2 sales hours of labor. A 40 inch television requires 18 cubic feet of space, and takes 3 sales hours of labor. A maximum of 1080 cubic feet of storage space is available. The store has a maximum of 198 hours of labor available. The profit earned from each of these sizes of televisions is \$60 and \$80, respectively.

(a) How many of each size of television should be sold to maximize the store's profit?

(b) What is the maximum profit?

(c) Are there any resources left over?

x = # of 21 in TVs

y = # of 40 in TVs.

Objective Fcn: Max $P = 60x + 80y$

Subject to: Storage space: $6x + 18y \le 1080 \Rightarrow 6x + 18y + s_1 = 1080$

labor hours: $2x + 3y \le 198 \Rightarrow 2x + 3y + s_2 = 198$

$x \ge 0$

$y \ge 0$

Std max prob?
1. obj fcn max ✓
2. non-neg variables ✓
3. variable stuff $\le$ non-neg # ✓

Initial Simplex Tableau

$x \quad y \quad s_1 \quad s_2 \quad P \mid$ constant

1080

$P = 60x + 80y$

Initial Simplex Tableau

x	y	S_1	S_2	P	constant
6	18	1	0	0	1080
2	3	0	1	0	198
-60	-80	0	0	1	0

$P = 60x + 80y$

$-60x - 80y + P = 0$

y & S_2 are non-basic variables

x, S_1, P are basic variables

Initial Simplex Tableau

x	y	s_1	s_2	P	constant
6	18	1	0	0	1080
2	3	0	1	0	198
−60	−80	0	0	1	0

$\frac{1080}{18} = 60 \leftarrow$ pr

$\frac{198}{3} = 66$

pc

Final Simplex Tableau

x	y	s_1	s_2	P	constant
0	9	1	−3	0	486
1	1.5	0	0.5	0	99
0	10	0	30	1	5940

$S_1 = 486$
$x = 99$
$P = 5940$

$x = 99$ $S_1 = 486$ $P = 5940$
$y = 0$ $S_2 = 0$

An optimal soln has been reached.
$\Rightarrow$ no negatives in last row to left of vertical line

(a) How many of each size of television should be sold to maximize the store's profit?

Should sell 99 21-inch TVs & 0 40-inch TVs

(b) What is the maximum profit?

Max Profit: $5,940

(c) Are there any resources left over? Yes.

486 cubic feet of storage space left over.

- A linear programming problem will have
  - no solution: When the simplex method breaks down (i.e. can find no non-negative ratios).

x	y	s_1	s_2	P	constant
0	−5	4	−1	0	35
1	0	0	1	0	22
0	−5	0	3	1	18

$\frac{35}{-5} = -7$ X
$\frac{22}{0}$ X

need non-negative ratios

Cannot find pivot element

Cannot divide by − or by 0 when finding pivot row

pc

No soln

an optimal soln has not been reached.

  - Infinitely many solutions: When the last row to the left of the vertical line has a 0 in a column that is not a unit column.

x	y	z	s_1	s_2	P	constant
1	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	30
0	0	$-\frac{27}{5}$	1	$-\frac{3}{5}$	0	10
0	0	$\frac{26}{5}$	0	$\frac{2}{5}$	1	60

$x + y = 30 \Rightarrow x = 30 - y$, $y = y \Rightarrow$ Let $y = t$, any real #
$x = 30 - t$
$y = t$

$S_1 = 10$

$P = 60$

An optimal soln has been reached

$x = 30 - t$ $S_1 = 10$ ... many solns

An optimal soln has been reached

$x = 30 - t \quad s_1 = 10$

$y = t \quad s_2 = 0$

$z = 0 \quad p = 60$

Inf. many solns

Ex: Pies Galore specializes in chocolate cream, lemon meringue, and tart cherry pies. Each chocolate cream pie uses 1 pie crust, 1 serving of whipped cream, and 3 servings of sugar. Each lemon meringue pie uses 1 pie crust, 2 servings of whipped cream, and 3 servings of sugar. Each tart cherry pie uses 2 pie crusts, 1 serving of whipped cream, and 1 serving of sugar. Pies Galore has not received a food shipment in a while and only has 50 pie crusts, 100 servings of whipped cream, and 120 servings of sugar on hand. They sell each chocolate cream pie for \$15, each lemon meringue pie for \$14, and each tart cherry pie for \$20. How many of each type of pie should Pies Galore make and sell in order to maximize their revenue, given their current inventory? Does Pies Galore have any ingredients leftover, when maximizing their revenue?

x = # of choc. cream pies

y = # of lemon meringue pies

z = # of tart cherry pies

Obj fn: Max $R = 15x + 14y + 20z$

Subject to:

pie crust: $x + y + 2z \le 50$ $+ s_1 =$

cream: $x + 2y + 1z \le 100$ $+ s_2 =$

sugar: $3x + 3y + z \le 120$ $+ s_3 =$

$x \ge 0$

$y \ge 0$

$z \ge 0$

Bottom

Initial Simplex Tableau

x	y	z	s_1	s_2	s_3	R	Constant
1	1	2	1	0	0	0	50
1	2	1	0	1	0	0	100
3	3	1	0	0	1	0	120
−15	−14	−20	0	0	0	1	0

Initial Simplex Tableau

x	y	z	s_1	s_2	s_3	P	constant
1	1	2	1	0	0	0	50
1	2	1	0	1	0	0	100
3	3	1	0	0	1	0	120
−15	−14	−20	0	0	0	1	0

PC

$\frac{50}{2} = 25 \leftarrow$ pr

$\frac{100}{1} = 100$

$\frac{120}{1} = 120 \rightarrow \rightarrow \rightarrow$

Final Simplex Tableau

x	y	z	s_1	s_2	s_3	P	constant
1	1	0	−0.2	0	0.4	0	38
0	0	1	0.6	0	−0.2	0	6
0	1	0	−0.4	1	−0.2	0	56
0	1	0	9	0	2	1	690

$x = 38$, $z = 6$, $s_2 = 56$, $P = 690$

$x = 38 \quad z = 6 \quad s_2 = 56 \quad p = 690$

$y = 0 \quad s_1 = 0 \quad s_3 = 0$

(a) How many of each type of pie should Pies Galore make and sell in order to maximize their revenue, given their current inventory?

(a) How many of each type of pie should Pies Galore make and sell in order to maximize their revenue, given their current inventory?

Should make: 38 choc cream pies
0 lemon meringue
6 cherry pies

(b) What is the largest revenue it can realize?

$690

(c) Does Pies Galore have any ingredients leftover, when maximizing their revenue? yes.

56 servings of cream leftover