• Up to now, we have integrated over an interval \([a, b]\). Now we will integrate over a curve \(C\) using line integrals.

• Suppose a plane curve \(C\) is given by \(x = x(t), \ y = y(t), \ a \leq t \leq b\) (or by \(\mathbf{r}(t) = x(t)
\mathbf{i} + y(t)\mathbf{j}\)) and assume \(C\) is smooth (i.e. \(\mathbf{r}'\) is continuous and \(\mathbf{r}'(t) \neq \mathbf{0}\)).

• We can divide the curve \(C\) into subarcs, sum them up, and take the limit.

• **Definition:** If \(f\) is defined on a smooth curve \(C\) then the line integral of \(f\) along \(C\) is

\[
\int_C f(x, y) \, ds
\]

• **Recall:** Arc length of \(C\):
\[
L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
\]

Also, \(ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt\).

• So, \(\int_C f(x, y) \, ds = \)

• **Recall:** Arc length is independent of parametrization. Thus the value of a line integral is also independent of parametrization, as long as

• Special case: If \(C\) is the line segment from \((a, 0)\) to \((b, 0)\). Here, \(x\) is the parameter. Then the parametric equations are

So \(\int_C f(x, y) \, ds = \)

Ex: Evaluate \(\int_C xy^4 \, ds\) where \(C\) is the right half of the circle \(x^2 + y^2 = 16\).
Suppose $C$ is a piecewise smooth curve

Ex: Evaluate $\int_C x\,ds$ where $C_1$ is the line $y = x$ from $(0,0)$ to $(1,1)$ followed by $C_2$ which is the piece of the parabola $y = x^2$ from $(1,1)$ to $(0,0)$. 
• **NOTE:** Can use any parametrization that produces the curve.

• **Recall:** “linear” mass = $\int$ density $ds$.

• Can use line integrals to find the mass $m$ of a wire – this time shaped like a curve $C$. (Use $ds$ instead of $dA$ => now looking at linear density, not area density.)

Ex: A wire takes the shape of a semicircle $x^2 + y^2 = 1, y \geq 0$ and is thicker near its base than its top. Find the center of mass of the wire if linear density at any point is proportional to its distance from the line $y = 1$. 
\[ \int_C f(x,y)ds \] is the line integral with respect to arc length.

- Line integrals with respect to \( x \) and \( y \):
  \[ \int_C f(x,y)dx = \int_C f(x,y)dy = \]

- Can express everything in terms of \( t \): Let \( x = \)
  \[ \int_C f(x,y)dx = \quad \text{and} \quad \int_C f(x,y)dy = \]

- So, \( \int_C f(x,y)dx = \text{and} \int_C f(x,y)dy = \)

- If we have the line integral with respect to \( x \) and \( y \) together, we can abbreviate:
  \[ \int_C P(x,y)dx + \int_C Q(x,y)dy = \]

- The hardest part is usually finding the parametrization.

- **Recall:** A line segment starting at \( \vec{r}_0 \) and ending at \( \vec{r}_1 \) is

  Ex: Evaluate \( \int_C ydx + x^2dy \) where \( C \) is a) the line segment from \((-1, -1)\) to \((2, 2)\).

  b) the arc of the parabola \( x = y^2 - 2 \) from \((-1, -1)\) to \((2, 2)\).
• **NOTE:** In a) and b), \( C \) went from \((-1, -1)\) to \((2, 2)\) but along different paths. The answers are different. Thus line integrals depend on

• **NOTE:** Line integrals also depend on direction (orientation) of the curve.

• A given parametrization \( x = x(t), y = y(t), a \leq t \leq b \) determines an orientation of a curve \( C \) where positive orientation corresponds to increasing \( t \).

• Let \(-C\) denote the curve \( C \) with opposite orientation. So \( \int_{-C} f(x,y)dx = - \int_C f(x,y)dx \) and \( \int_{-C} f(x,y)dy = - \int_C f(x,y)dy \) (i.e. reverse the limits)

• **NOTE:** If integrate with respect to arc length, the value of the line integral does NOT change.

• **Line Integrals in Space** (just add another dimension – same thing)

  Suppose \( C \) is a smooth space curve given by \( x = x(t), y = y(t), z = z(t) \) (or \( \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \)).

  \[
  \int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt
  \]

• **NOTE:** \( ds = |\vec{r}'(t)| \) because \( \vec{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \)

• Then \( \int_C f(x,y,z)ds = \int_a^b f(\vec{r}(t))|\vec{r}'(t)| \, dt \)
• Likewise, we can define line integrals with respect to $x$, $y$, and $z$.

Ex: Evaluate $\int_C y \sin z \, ds$, where $C$ is the circular helix given by $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$.

Ex: Evaluate $\int_C x^2 \, dx + y^2 \, dy + z^2 \, dz$ where $C$ consists of the line segment from $(0,0,0)$ to $(1,2,-1)$ followed by the line segment from $(1,2,-1)$ to $(3,2,0)$. 
• **Line Integrals over Vector Fields**

Recall: \( W = \int_a^b f(x)dx \) and \( W = \mathbf{F} \cdot \mathbf{D} \) where \( \mathbf{D} = \mathbf{PQ} \) displacement

• Suppose \( \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \) is a continuous force field on \( \mathbb{R}^3 \). We want to compute the work done by this force in moving a particle along a smooth curve \( C \).

• Divide \( C \) into subarcs with length \( \Delta s \).

• As the particle moves along \( C \), the direction of motion at any point is given by the unit tangent vector \( \mathbf{\tilde{T}} \) multiplied by \( \Delta s \).

• So, \( \mathbf{\tilde{D}} = \)

• Work: \( W = \)

• Recall:

• So \( W = \)

• **Definition:** Let \( \mathbf{\tilde{F}} \) be a continuous vector field defined on a smooth curve \( C \) given by the vector function \( \mathbf{\tilde{r}}(t) \), \( a \leq t \leq b \). Then the line integral of \( \mathbf{\tilde{F}} \) along \( C \) is

• (Recall: \( \mathbf{\tilde{F}}(\mathbf{\tilde{r}}(t)) \) means \( \mathbf{\tilde{F}}(x(t), y(t), z(t)) \) and \( d\mathbf{\tilde{r}} = \mathbf{\tilde{r}}'(t)dt \))

Ex: Find the work done by the force field \( \mathbf{\tilde{F}}(x,y) = x\sin y\mathbf{i} + y\mathbf{j} \) in moving a particle along the parabola \( y = x^2 \) from \((-1,1)\) to \((2,4)\).
NOTE: \[ \int_C \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r} \]

Ex: Evaluate \[ \int_C \vec{F} \cdot d\vec{r} \] where \( \vec{F}(x,y,z) = xy\hat{i} + yz\hat{j} + zx\hat{k} \) and \( C \) is the twisted cube given by \( x = t, y = t^2, z = t^3, \) \( 0 \leq t \leq 1. \)

- The connection between line integrals of vector fields and line integrals of scalar fields (just a scalar function):
  Let \( \vec{F} = P\hat{i} + Q\hat{j} + R\hat{k} \) be a vector field. Then \[ \int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz. \]